Charged magnetosonic solitons propagating in gentle density gradients and wave breaking

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A class of nonrelativistic magnetosonic soliton, where the charge-separation electric field can be as large as the background magnetic field, is discovered in a strongly magnetized cold plasma, where the electron cyclotron frequency is much larger than the electron plasma frequency. We study how the Mach number of such a soliton is changed by the presence of a gentle background density gradient. An effective Hamiltonian for the soliton trajectory is derived, with which one can show that the soliton Mach number increases as the soliton travels to a background of increasing density, or equivalently, decreasing Alfven speed. The soliton can generally exchange its energy, momentum, and mass with the background plasmas. Our results also show that the soliton may undergo wave breaking at a finite background Alfven speed. [S1063-651X(97)12901-4]

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I. INTRODUCTION

In the context of astrophysics, collisionless plasmas tend to occur in energetic environments where the plasmas are hot and particle collisions are rare. Examples include the neutron star magnetospheres, pulsar nebulas, extended extragalactic radio sources (jets), and others. These environments are believed to be so energetic that characteristic waves, such as Alfven waves, and flows all have speeds close to the speed of light. There is a variety of observational evidence substantiating the above assessments. For example, high-resolution radio maps show that bright blobs in the inner part of the extragalactic radio sources move at relativistic speeds away from the cores [1]. Another distinct example is the ''wisps'' in the Crab Nebula [2–6]. Optical images show that the wisps waive quasicoherently on a time scale of several years and a length scale of a couple of light years. No matter whether they are blobs or wisps, these observationally identifiable objects are some forms of large-amplitude coherent fluctuations. These phenomena sparked the recent surge of interest in the study of solitons and shock waves in relativistic collisionless plasmas [5–11].

Several years ago, an interesting class of relativistic solitons in magnetoplasmas was discovered [9,10]. They differ from solitons in an electron-positron pair plasma [12] in that ion inertia plays an important role in setting up charge separation within the solitons, whereas electron inertia can be ignored altogether; thus electrons serve only to provide an electrically neutralizing background. These solitons have typical length scales on the order of the ion gyroradius, much larger than that of the conventional magnetosonic solitons for which the electron inertia is of primary importance [12]. Although ideal solutions for solitons in isolation have been obtained, one probably needs to be somewhat more conservative in applying them to explain the observed moving objects directly, since the astrophysical environments are not as ideal as one assumes to obtain these solutions. Among all nonideal factors, we are most interested in the effects of gentle changes in the environment on the solitons. As solitons propagate they gradually enter new environments, and a question naturally arises as to whether the solitons may become more energetic or less energetic. Or, to put it another way, do they become more focused or less focused? If the solitons become more focused, will they reach a state of wave breaking?

This question is important not only because of the trivial energetic consideration, but because an ever-increasingly focused soliton can also be prone to coupling to dissipation, thereby heating the background plasmas which emit the observed bright light. This phenomenon is anticipated from a common experience that when shallow water waves approach the sea shore, the waves become steeper and steeper, and finally break up not far from the shore. It is well known that the existence of a soliton arises from a detailed balance between the wave dispersion and nonlinear focusing. Propagation of a soliton into a gradually changing environment must also affect such fine balance in some subtle ways. In certain circumstances, wave breaking analogous to that of the shallow water waves may occur. Our program of study to follow aims at an understanding of this aspect of solitons in relativistic magnetoplasmas. This series of studies begins with the present analytical work, that addresses a simpler problem for the propagation of a nonrelativistic soliton in a magnetized plasma with a gentle density gradient. An understanding of its fundamental mechanism will surely help us proceed to address the ultimate problem for the relativistic solitons.

This program requires us to construct a counterpart of the aforementioned class of relativistic solitons in the nonrelativistic regime. As we shall show in the following sections, the procedure of taking $c\to\infty$ for our relativistic soliton solutions must be carried out with caution since certain approximations, such as ignoring the electron inertia, which can be a good approximation for the relativistic regime but may not necessarily be so in the nonrelativistic regime, have been adopted. For example, one must address which of the quantities ω_{pe}, the electron plasma frequency, and ω_{ce}, the electron gyrofrequency, is larger in the nonrelativistic case. Since $\omega_{ce}^2/\omega_{pe}^2=m_iV_A^2/m_ec^2$, where V_A and c are, respectively, the Alfven speed and light speed and m_i and m_e the ion mass and electron mass, there is no such problem in the relativistic regime where $V_A\to c$. However, it becomes a problem in the nonrelativistic limit, in that the limit $c\to\infty$ must confront the neglect of the electron inertia ($m_e\to 0$). In fact, careful scrutiny shows that, for the present analyses to be self-consistent, one must choose the regime that $\omega_{pe}/\omega_{ce}\rightarrow 0$, suggesting that we are confined to strongly magnetized plasmas, which are likely to exist in astronomical environments, such as at the active galactic nuclei [13]. Since the soliton speed is greater than the Alfven speed, this strongly magnetized regime also implies that it is in the energy range well above 1 MeV, but still less than 1 GeV due to the nonrelativistic treatments for ions. Thus this is the natural extension from the fully relativistic regime to the nonrelativistic regime. For simplicity, we shall also assume that both electrons and ions are cold.

Before beginning the analyses, we want to make some approximate arguments justifying why such a soliton should exist. When $\omega_{ce}\gg\omega_{pe}$, electrons are strongly tied to the field lines, and any movement of electron guiding centers across the magnetic fields will have to drag the field lines along; in this sense, electrons appear much heavier than the ions in serving as a relatively rigid neutralizing background. When the flow enters the soliton where the electrostatic potential is positive, the ions are immediately retarded from the potential well, whereas the electrons continue to march ahead together with the field lines. These overshooting ions eventually are pulled back with the magnetized flow because of the restoring force. Charge separation due to the demagnetized ion motion must in turn sustain the needed electrostatic potential self-consistently. (Of course, the magnetized electrons are not infinitely heavy, and therefore their guiding centers in fact are mildly retarded in response to the pull of the ions, thereby yielding a local pileup of field lines.) It is obvious from the above arguments that the length scale of the soliton must be of the order of the ion scale. On the other hand, the above picture will have to break down if electrons are not strongly tied to the field lines, i.e., $\omega_{ce}\ll\omega_{pe}$. In this regime, the electrons move more freely across the field lines, and can respond to the electric force more sensitively. Therefore, not only can electrons easily shield out the charge-separation electric field, but the electromagnetic effects can become of primary importance. In this weakly magnetized regime, electrons play a prominent role, and hence the length scale of interest is expected to be of the electron inertia length scale c/ω_{pe} [14,15]. (A further elaboration on this issue is given in the Appendix.)

This paper is organized as follows. Section II constructs the ideal solution for a nonrelativistic soliton in isolation. An action principle is proposed to examine the dynamics of the soliton in a changing environment in Sec. III. This proposal yields an effective Lagrangian of a point particle representing the spatial integral of the soliton. Armed with this result, the soliton trajectory for any given static background density profile is solved in the form of the integral of motion. We give discussions and conclusions in Sec. IV.

II. NONRELATIVISTIC MAGNETOSONIC SOLITONS

The physical condition to be considered consists of a background magnetized plasma with the magnetic field **B** in the z direction and a compressional wave propagating in the x direction. The gentle background density gradient is in the direction of wave propagation. For the construction of an ideal soliton solution, we may ignore the density gradient for the time being. Associated with the wave are not only the compressional density and magnetic fluctuations, but also velocity fluctuations of the ions and electrons in both x and y directions. These velocity fluctuations are responsible for the sources of the charge-separation electric field and magnetic field fluctuations. We may denote u_j and v_j as the x and y components of the velocity, respectively, and the subscript j denotes the index for the species which can be either ion or electron.

In the following analysis, we assume that the absolute value of the charge for the ion is the same as that of the electron for the sake of simplicity. Let $|q_j|=e$. It is straightforward to relax this assumption to a general case.

In the wave rest frame, the problem becomes a time-independent one, and we have the following governing equations. The continuity equation for either species reads

$$\frac{d}{dx}(n_j u_j)=0. \tag{1}$$

The momentum equations for either species are

$$m_j u_j \frac{du_j}{dx}=q_j\left(E_x+\frac{v_j}{c}B_z\right) \tag{2}$$

and

$$m_j u_j \frac{dv_j}{dx}=q_j\left(E_y-\frac{u_j}{c}B_z\right). \tag{3}$$

The relevant Maxwell's equations are

$$\frac{dB_z}{dx}=-\frac{4\pi}{c}\sum_j q_j n_j v_j, \tag{4}$$

$$\frac{dE_y}{dx}=0, \tag{5}$$

and

$$\frac{dE_x}{dx}=4\pi\sum_j q_j n_j. \tag{6}$$

Equation (5) immediately gives

$$E_y=E_0=\text{const.} \tag{7}$$

Before manipulating further algebra, we turn to a discussion of the physical quantities, which a subscript ''0'' denotes, in the uniform far upstream region. There is no electric potential gradient far upstream, and hence the electric field becomes

$$\mathbf{E}_0=E_0\hat{\mathbf{y}}. \tag{8}$$

Together with this relation, Eq. (2) yields that

$$v_{j0}=0, \tag{9}$$

Eq. (3) yields

$$u_{j0}=c\frac{E_0}{B_0}\equiv u_0, \tag{10}$$

the $E\times B$ drift, and Eq. (6) yields

$$n_{i0}=n_{e0}\equiv n_0. \tag{11}$$

It is a straightforward matter to construct the conserved fluxes. Besides Eq. (1), that represents the conservation of mass flux for each species, which yields

$$n_j u_j = n_0 u_0, \tag{12}$$

we also have a conserved energy flux

$$T^{01}=\tfrac{1}{2}\sum_j n_o u_0[m_j(u_j^2+v_j^2)]+\frac{c}{4\pi}E_0B_z, \tag{13}$$

where the second term on the right is the Poynting flux, and there are conserved momentum fluxes in the x and y directions, respectively:

$$T^{11}=\sum_j n_0u_0(m_ju_j)+\frac{1}{8\pi}(B_z^2+E_0^2-E_x^2), \tag{14}$$

and

$$T^{21}=\sum_j n_0u_0(m_jv_j)-\frac{1}{4\pi}E_0E_x=0. \tag{15}$$

The second equality of Eq. (15) results from the vanishing v_{j0} and E_{x0} far upstream.

Analogous to what was discovered in the previous relativistic case, we may also construct another conservation law for a time-independent problem in the nonrelativistic regime. Multiplying the x component of the momentum equation, Eq. (2), by m_jq_j/e, and summing up the resultant equations for both species, we find that

$$\frac{1}{2}\frac{d}{dx}[(m_iu_i)^2-(m_eu_e)^2]=eE_x\left[m_i+m_e+\frac{E_0B_z}{4\pi n_0u_0c}\right]$$
$$=eE_x\left[\sum_j m_j\left(1-\frac{u_j^2+v_j^2}{2c^2}\right)+\frac{T^{01}}{n_0u_0c^2}\right], \tag{16}$$

where we have employed Eq. (15) to obtain the first equality, and Eq. (13) to obtain the second equality. For a nonrelativistic formulation where we have already ignored terms of order $\mathbf{v}_j^2/c^2$, terms of this order appearing in Eq. (16) should have been dropped for the sake of self-consistency. Beside the obvious terms in the right side of the second equality of Eq. (16), we may examine the term proportional to T^{01} in Eq. (16). This term is also of order $\mathbf{v}_j^2/c^2$ (or V_A^2/c^2) compared with the remaining term, and thus we drop it. Also ignoring m_e in comparison with m_i, we finally reduce the right side of Eq. (16) to em_iE_x. Let $E_x\equiv -d\phi/dx$, and we arrive at the conservation law

$$Q=\tfrac{1}{2}(m_i^2u_i^2-m_e^2u_e^2)+m_ie\phi. \tag{17}$$

Thus far, we have completed our construction of conservation laws for such a dynamical system, to be followed by some algebraic manipulations for deriving a useful soliton equation.

We proceed to assume that the electrons are frozen in the field lines by ignoring the electron inertia in Eqs. (2) and (3). As it will become clear that the change of B_z is of order the background field B_0, the electron drift $u_e(=cE_0/B_z)$ is therefore of order $u_0(=cE_0/B_0)$. In addition, the ion x-component velocity u_i will also be of order u_0, so it hence follows that $m_eu_e^2\ll m_iu_i^2$ and $m_eu_e\ll m_iu_i$. These orderings of terms take effects on Eqs. (14) and (17), and also on the x component of the kinetic energy in Eq. (13), where these electron contributions can be neglected.

Next we compare the transverse momentum fluxes of ions and electrons in Eq. (15). If the ion transverse momentum flux dominates and we ignore the electron contribution, it follows that

$$v_i=E_0E_x/4\pi m_in_0u_0. \tag{18}$$

Since the electrons undergo the $\mathbf{E}\times\mathbf{B}$ drift, we have

$$v_e=m_ecE_x/m_eB_z. \tag{19}$$

Comparing the two transverse momenta, we find that $m_iv_i/m_ev_e\sim O(m_iV_A^2/m_ec^2)\sim O(\omega_{ce}^2/\omega_{pe}^2)$. As mentioned in Sec. I, we shall fix our parameters in the strong magnetic field regime, and therefore the above ratio is much greater than unity; the ion transverse momentum flux indeed dominates in Eq. (15), indicating the validity of the $\mathbf{E}\times\mathbf{B}$ electron drift motion.

With all the above considerations, we may solve for B_z from Eq. (13), with u_i expressed in terms of ϕ from Eq. (17), and v_i and v_e in terms of E_x from Eqs. (18) and (19). Finally, as we substitute this B_z into Eq. (14) and again express u_i in terms of ϕ, we obtain an equation that almost relates E_x with ϕ,

$$\frac{1}{B_0^2}\left(\frac{d\phi}{dx}\right)^2+1-\left(\frac{4\pi en_0}{B_0^2}\phi-\frac{1}{2B_0^2}\left(\frac{V_{A0}^2}{2c^2}+\frac{m_ec^2}{m_iV_A^2}\right)\left(\frac{d\phi}{dx}\right)^2+1\right)^2-\frac{8\pi m_in_0u_0^2}{B_0^2}\left[\left(1-\frac{2e\phi}{m_iu_0^2}\right)^{1/2}-1\right]=0. \tag{20}$$

The third term in Eq. (20) contains those that are proportional to $(d\phi/dx)^2$. They have small coefficients in comparison with the first term in Eq. (20), as we are confined to a nonrelativistic $(V_A^2/c^2\ll 1)$ and strongly magnetized $(m_ec^2/m_iV_A^2=\omega_{pe}^2/\omega_{ce}^2\ll 1)$ plasma. One may trace the origins of these negligibly small terms, and find that they come from the ion and electron transverse kinetic energies, $m_iv_i^2/2$ and $m_ev_e^2/2$, respectively, in Eq. (13). Thus not only does the electron inertia completely drop out in the final equation, but the ion transverse energy flux also drops out.

Rearranging Eq. (20) and dropping the undesired terms, we finally arrive at the soliton equation

$$\frac{1}{2}\left(\frac{d\Phi}{d\eta}\right)^2+U(\Phi)=0, \tag{21}$$

where

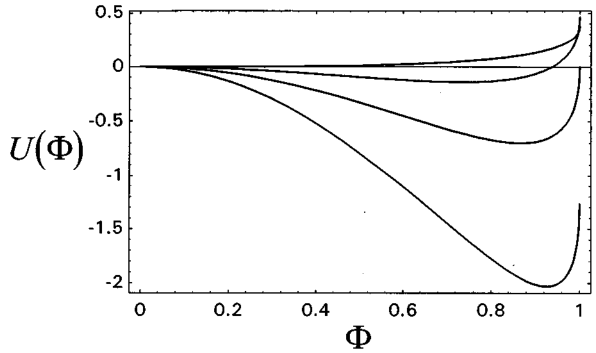

FIG. 1. Typical ''pseudopotentials'' $U(\Phi)$ in Eq. (22) for the soliton solutions. Plotted are the four $U(\phi)$'s for $M_A=1$, 1.6, 2, and 2.4 in sequence beginning from the top.

$$U(\Phi)\equiv\frac{M_A^2}{2}\left((1-\sqrt{1-\Phi})^2-\frac{M_A^2}{4}\Phi^2\right). \qquad (22)$$

$\Phi\equiv 2e\phi/m_i u_0^2$, $\eta\equiv 2eB_0x/m_iu_0^2$, and $M_A\equiv u_0/V_A$ is the Alfven Mach number. Note that the length scale of the solution is about V_A/ω_{pi} or $r_cV_A/c \ll r_c$, where r_c is the ion gyroradius using u_0 as the typical speed. The solution Φ oscillates in the ''potential well'' $U(\Phi)$ from one zero of U to the other.

Plotted in Fig. 1 are $U(\Phi)$ for various Mach numbers M_A. For small Φ, the ''potential'' $U(\Phi)\approx -M_A^2(M_A^2-1)\Phi^2/8$, yielding a decaying solution $\Phi\approx\exp(-M_A\sqrt{M_A^2-1}|\eta|/2)$ at large distances. The behavior shows that the solution is a soliton. When $M_A=2$, the ''potential'' $U(\Phi)$ becomes singular at $\Phi=1$, where $U(\Phi)=0$ and the solution captures a singularity. This singular point occurs at the far side zero of U, corresponding to a cusp at the soliton peak, where coupling to collisionless dissipation becomes inevitable and the soliton must break. On the other hand, a soliton of a moderate strength has a typical $E_x=O(B_0)$, which is much greater than E_0 by a factor c/V_A. That is, this soliton is dominantly sustained by the charge separation rather than the electric current alone.

We may expand $U(\Phi)$ further for a small Φ and keep up to the Φ^3 term. One finds

$$U(\Phi)=-\tfrac{1}{8}M_A^2(M_A^2-1)\Phi^2+\frac{M_A^2}{16}\Phi^3+O(\Phi^4). \qquad (23)$$

Up to this order, the soliton solution can be expressed in an analytical form,

$$\Phi\approx 2(M_A^2-1)\mathrm{sech}^2\left(\frac{M_A\sqrt{(M_A^2-1)}}{4}\,\eta\right). \qquad (24)$$

This small-amplitude soliton solution can only be self-consistent when $M_A-1\ll 1$, and this expression is identical to the Korteweg-de Vries soliton of shallow water. The amplitude scales as M_A-1, and the length as $1/\sqrt{M_A-1}$. To be complete, we list relevant physical quantities expressed in terms of Φ in this small-amplitude regime,

$$u_i=u_0\left(1-\frac{\Phi}{2}\right),\quad n_i=n_0\left(1+\frac{\Phi}{2}\right),$$

$$B_z=B_0\left(1+\frac{\Phi}{2}\right),\quad E_x=-B_0\frac{d\Phi}{d\eta}. \qquad (25)$$

In addition, we may examine what type of linear waves they are in association with such solitons by dropping the Φ^3 term in Eq. (23). Differentiating both sides of Eq. (21) with respect to x, we take a spatial Fourier transform, $d/dx\rightarrow ik$, and let the phase velocity $\omega/k=u_0$. We obtain the dispersion relation $\omega^2/k^2=V_A^2/(1+k^2V_A^2/\omega_{pi}^2)$, which recovers the compressional Alfven wave at long wavelengths. (Also see the Appendix.)

Thus far, we may cast the question this paper intends to address in the following terms. Given the soliton derived above propagating in a gentle density gradient, will its Mach number increase or decrease? If the former, the soliton will steepen and is prone to collisionless dissipation; if the latter, the soliton will become more dispersive. In the following section, this question will be answered.

III. ACTION PRINCIPLE AND INTEGRAL OF SOLITON MOTION

In the presence of a gently varying static background, we may assume that the soliton remains as a localized disturbance, not much different from the ideal solution constructed in Sec. II. If we let the ratio of the soliton width to the typical length scale of the background be a small parameter ϵ, the soliton shape and the Mach number may both change by an amount of order ϵ. In spite of these changes of order ϵ, we assume that the action for the soliton is form invariant, as a functional integral of Φ [cf. Eqs. (27) and (28) below]. As long as the form of the action is preserved, the construction of the soliton trajectory in the following analysis depends little on how the soliton shape becomes modified.

Since the soliton can be either accelerated or decelerated, it is only appropriate to work out this program in the laboratory frame of reference, where the background is at rest, rather than in the soliton frame as carried out in Sec. II. This requires us to transform all physical quantities derived above to the laboratory frame before the construction of the Lagrangian. In the relativistic framework, the Lagrangian density is Lorentz invariant. In the nonrelativistic framework, the electromagnetic part of the Lagrangian density, $(\mathbf{E}^2-\mathbf{B}^2)/8\pi$, is frame independent, the particle part $m_jn_j\mathbf{v}_j^2/2$ should be changed to $m_jn_j(\mathbf{v}_j-u_0\hat{x})^2/2$ when changing to the laboratory frame, and the interaction part $\Sigma(q_jn_jv_jA_y/c-q_jn_j\phi)$ also remains the same. Moreover, we must also ignore some minor terms in the particle kinetic energy, such as the electron inertia and $m_iv_i^2/2$ in the Lagrangian density, in order to be consistent with the expression of T^{01} used in deriving the soliton. It follows that the total Lagrangian in the laboratory frame becomes

$$L=\int_{-\infty}^{\infty}\left(\frac{1}{2}m_in_i(u_i-u_0)^2-\frac{1}{8\pi}(B_z^2-E_x^2-E_0^2)\right.$$
$$\left.+\frac{1}{4\pi}\phi\frac{d^2\phi}{dx^2}-\frac{1}{4\pi}A_y\frac{dB_z}{dx}\right)dx. \qquad (26)$$

Certain technicalities must be mentioned before computing L. That is, the last term in the integral involves the vector potential A_y, which does not appear in our previous analysis. To handle this term, we note that

$$\int_{-\infty}^{\infty} A_y \frac{dB_z}{dx} dx = A_y B_z|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} B_z^2 dx$$
$$= \int_{-\infty}^{\infty} (B_0 B_z - B_z^2) dx.$$

The second equality arises from another notion that when $x = \pm\infty$, $B_z = B_0$ and

$$A_y|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} B_z dx.$$

A straightforward but somewhat tedious calculation shows that the total action can be cast into the form,

$$S = \int L\, dt, \tag{27}$$

where

$$L = -\int_{-\infty}^{\infty} \frac{B_0^2}{16\pi} dx - \frac{B_0^3 M_A^3}{8\pi^2 e n_0} \int_0^{\Phi_{\max}} \Phi \frac{1 - 1/\sqrt{1-\Phi}}{[M_A^2 \Phi^2 - 4(1-\sqrt{1-\Phi})^2]^{1/2}} d\Phi$$
$$= L_0 + \Delta L(M_A, n_0), \tag{28}$$

where L_0 represents the background Lagrangian, ΔL the soliton Lagrangian, and $\Phi_{\max} \equiv 4(M_A - 1)/M_A^2$. In Eq. (28), we deliberately express all parameters in ΔL in terms of M_A, n_0, and B_0 for the sake of convenience when used later. Since B_0 is uniform, the dependence on M_A and $n_0(x)$ is equivalent to dependence on u_0 and x; that is, analogous to the classical particle dynamics, the relevant Lagrangian is a function of the soliton velocity u_0 and position x.

Retracing what $1 - (1/\sqrt{1-\Phi})$ means in Eq. (28), we find that $(1/\sqrt{1-\Phi}) - 1 \propto \Delta n_i \equiv n_i - n_0$, and that the Lagrangian of the soliton is nothing more than the kinetic energy of an ion density excess Δn_i,

$$\Delta L = \tfrac{1}{2} \int_{-\infty}^{\infty} m_i u_0^2 \Delta n_i dx. \tag{29}$$

This is a rather surprising result, in that all other contributions from the particle velocity variations and field variations associated with the soliton are completely canceled as if the soliton were a localized and structureless density excess in motion.

Having the Lagrangian, we are in a position to construct the equation of motion using the action principle

$$\delta S = \int \delta[L_0 + \Delta L(u_0, x)] dt = 0. \tag{27$'$}$$

In principle, with the above rather general formulation, one is not restricted to a static background for determining the equation of motion of the soliton. For a slowly time-dependent background, either one may treat the background as a given function of x and t, or, to be self-consistent, one may also consider the mutual interactions between the background and soliton using this Lagrangian formulation. For the former with a given background, one need not consider L_0, but, for the latter, the full Lagrangian must be taken into account as shown below.

Variation of L_0 yields the equation of motion for the background, which is, in the present case, reduced to the force balance equation

$$\frac{dB_0^2}{dx} = 0, \tag{30}$$

or B_0=const. Without particle pressure, indeed only in a uniform magnetic field can the background maintain force balance.

Variation in ΔL, however, presents some technical difficulty in obtaining an analytical result. This is because Φ is some complicated function of η and M_A which has no analytical expression let alone the integral of Eq. (28). Hence the equation of motion can only be constructed numerically.

A way to circumvent this technical difficulty and to proceed further is to look for the integral of motion. Fortunately, this is possible because the soliton propagates in a static background, and the Lagrangian therefore has no explicit time dependence. The constant of motion is the Hamiltonian $H(u_0, n_0(x))$. Using Eqs. (21) and (22), it is a straightforward matter to obtain an integral expression for the Hamiltonian, defined to be

$$H \equiv u_0 \frac{\partial L}{\partial u_0} - L. \tag{31}$$

Thus one finds that

$$H(M_A, n_0) = \int_{-\infty}^{\infty} \frac{B_0^2}{8\pi} dx + \frac{B_0^3 M_A}{16\pi^2 e n_0} \int_0^{\Phi_{\max}} \times \frac{N(M_A, \Phi)}{[M_A^2 \Phi^2 - 4(1-\sqrt{1-\Phi})^2]^{1/2}} d\Phi$$
$$= H_0 + \Delta H(M_A, n_0), \tag{32}$$

where

$$N(M_A,n_0)\equiv M_A^2\left(2\Phi+\frac{4(2-M_A)}{M_A^2}\right)\left(\frac{1}{\sqrt{1-\Phi}}-1\right)+\frac{2(2-M_A)\Phi}{(1-\Phi)^{3/2}}$$
$$+\frac{(M_A^4\Phi^2+4(2-M_A)(M_A^2\Phi+4)\Phi(1-1/\sqrt{1-\Phi})+8(2-M_A)\Phi^2/(1-\Phi)}{M_A^2\Phi^2-4(1-\sqrt{1-\Phi})^2}. \tag{33}$$

Here H_0 and ΔH are the corresponding Hamiltonians for the background and the soliton, respectively. The exact functional form of $\Delta H(M_A,n_0)$ has to be determined from numerical integration. Once ΔH is determined, we will have basically fulfilled our goal for this work, in that given an initial soliton of Mach number $M_A(t=0)$, when it enters a background of different density, we can obtain a different value of M_A by looking up to the constant contours of $\Delta H(M_A,n_0)$.

Plotted in Fig. 2 with the bold line is a contour of constant $\Delta H(M_A,n_0)$. The general trend is that M_A increases with increasing $n_0(x)$ or decreasing background Alfven speed $V_{A0}(x)$, and the soliton becomes steeper as a result. Because of several singular terms in the integrand of Eq. (32), it may, at first glance, seem that the integral should diverge at $M_A=2$ when $\Phi_{\max}=1$. These infinities, in fact, turn out to completely cancel, and a finite total integral is obtained.

Also plotted in Fig. 2 with the thin lines are the contours of constant total energy contained within the soliton in the laboratory frame,

$$\Delta W\equiv\int_{-\infty}^{\infty}\Delta T^{00\prime}dx=\frac{B_0^3M_A^3}{32\pi^2en_0}\int_0^{\Phi_{\max}}$$
$$\times\frac{4\Phi+6\sqrt{1-\Phi}-8+M_A^2\Phi^2+2/\sqrt{1-\Phi}}{[M_A^2\Phi^2-4(1-\sqrt{1-\Phi})^2]^{1/2}}\,d\Phi. \tag{34}$$

The upper thin line has 1.5 times as much energy as the lower thin line. As expected, the background energy density can be found to be identical to the background Hamiltonian density. What is somewhat unexpected is that the constant ΔH contour does not coincide with any of the constant ΔW contours. Thus the soliton energy is not conserved as it moves up, or down, the density gradient. In fact, the soliton moves toward the higher energy configurations as the background density, or the Mach number, increases. One may further find that neither the soliton momentum nor the soliton ''mass'' (total number of particles within the soliton) are conserved. These findings may seem somewhat unusual. However, if one realizes that the background can serve as a reservoir for the soliton to exchange energy, momentum, and even mass, these results would not be so surprising. As the soliton passes through an inhomogeneous medium, the downstream background left behind by the soliton may be modified by a small amount of order ϵ. Accumulation of many changes of order ϵ, after the soliton travels a long distance, can eventually produce a finite change in the soliton energy, momentum, and mass.

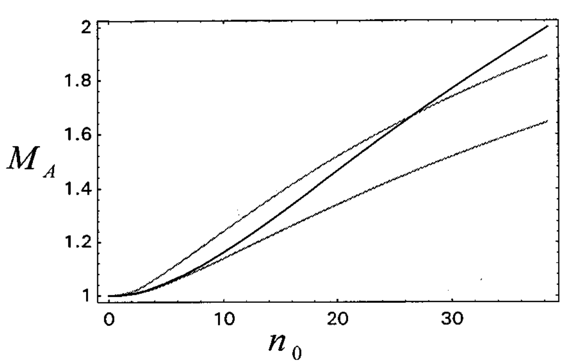

FIG. 2. The contour (heavy line) of constant $\Delta H(M_A,n_0)$ in the (M_A,n_0) phase space, and the two contours (shallow lines) of constant ΔW in the same phase space; the lower thin line assumes a value of ΔW, the same as ΔH, and the upper thin line a value 1.5 times as large as ΔH.

In addition, we may take advantage of the soluble regime of small-amplitude solitons where $M_A-1\ll 1$, and examine how ΔH should behave. We expand ΔL of Eq. (28) in the limit of small M_A-1 and Φ, and then use Eq. (31) to derive the Hamiltonian for small-amplitude solitons. Replacing the soliton solution from that in Eq. (24), one obtains that

$$\Delta H\propto\frac{B_0^3}{n_0e}\,(M_A-1)^{1/2}+O((M_A-1)^{3/2}). \tag{35}$$

In this small-amplitude limit, along the constant ΔH, the soliton Mach number changes as $M_A-1\propto n_0^2$.

To understand this result, by which the KdV-type soliton abides, we may look into it from another perspective. The small-amplitude limit of the soliton energy reads

$$\Delta W\propto\frac{B_0^3}{n_0e}\,(M_A-1)^{1/2}+O((M_A-1)^{3/2}), \tag{36}$$

after one carries out similar procedures. Note that ΔH and ΔW differ only by a constant in this limit, indicating that the constant energy contours track the constant Hamiltonian contours. The soliton energy is conserved regardless of whether the small-amplitude soliton propagates in an inhomogeneous or homogeneous background. Hence we conclude that it is only for a finite-amplitude soliton that exchange of energy with the background is possible.

IV. DISCUSSIONS AND CONCLUSIONS

In a series of works to follow, we aim at a thorough understanding of how solitons, in particular relativistic solitons, travel in a slowly varying environment. Specifically, we are interested in how and to what extent the soliton can become more and more focused as it travels through such an environment. As a first step toward this goal, we study a simpler case in the present work, where the soliton travels at a nonrelativistic speed in a cold and strongly magnetized collisionless plasma. A magnetosonic soliton solution in such a plasma is constructed, and the validity of the solution only holds when $\omega_{ce} \gg \omega_{pe}$. A soliton of this type is much like a moving capacitor, charged with an electric field comparable to the background magnetic field. Although it also carries a self-consistent electric current to keep the background magnetic field compressed, the soliton consists primarily of an electrostatic disturbance. This class of solitons differs significantly from the conventional magnetosonic solitons, for which the electron inertia plays an important role and the charge-separation electric field becomes negligibly small [14,15].

These conventional solitons of an electron inertia length scale are primarily of electromagnetic disturbances, and can only exist in the opposite regime to what is under consideration in the present work. (Consult the Appendix for details.) In fact, if we compare the Alfven ion length V_A/ω_{pi} with the electron inertia length c/ω_{pe}, we find that their ratio $V_A\omega_{pe}/c\omega_{pi}$ precisely equals $\sqrt{m_i V_A^2/m_e c^2} = \omega_{ce}/\omega_{pe}$. This ratio is much greater than unity in the present regime of interest, but, in the opposite regime, the electron inertia length c/ω_{pe} turns out to become much greater than the Alfven ion length V_A/ω_{pi}. At any rate, the solitons of either regime always choose the greater length scale of the two.

The analytical strategy, with which we tackle the problem of soliton propagation in a nonuniform plasma, is first to assume that the soliton retains its integrity throughout the journey. (In strict mathematical terms, this statement means that the soliton action remains form invariant.) We then treat the soliton as a quasiparticle for determining its mean trajectory. Although the equation of motion turns out to be rather complicated, we have, nonetheless, taken advantage of the *static* background and been able to construct an integral of motion. With the help of this integral of motion, one is able to answer the question as to how and to what extent the soliton can become more and more focused in a nonuniform environment. We find that it is only when the soliton propagates into an ever-increasing density region that the soliton can become steepened and the Mach number increased.

From Fig. 2, one notes that starting from a very small-amplitude soliton where $M_A \to 1$, the background density needs to increase by many folds in order for the soliton to become sufficiently steepened. However, if one starts with a soliton of moderate strength with a Mach number of about 1.5, it requires the background density to increase to about twice as high to reach a state of wave breaking at $M_A = 2$. The possibility of wave breaking can have important implications in solving one of the important problems in high-energy astrophysics. Many astrophysical objects emit synchrotron radiation in the radio wave band. The typical magnetic field strength in environments such as the extragalactic jets and the Crab Nebula is about mG, and to reach the radio frequency of GHz, it requires an electron energy about 1 GeV or so. Furthermore, the relativistic electrons lose energy very rapidly. Therefore it remains to be explained as to how these high energy electrons are replenished [8]. One possible solution entails solitons that are continuously initiated at some core regions and break at distance as they propagate down the road where the background Alfven speed decreases [10]. Unlike shock waves which lose energy on their way out, and hence cannot travel for long distances, the solitons can dump their energy afar with vigor. This may explain the observed bright blobs far from the obvious energy sources at the compact cores [1–3].

In fact, in our formulation of the problem, we deliberately ignored the plasma temperature and let the background magnetic field be uniform. This choice has the advantage of having an exact equilibrium background far from the soliton, and isolating the parameter of nonuniformity to the plasma density $n_0(x)$. Our result shows that the soliton Mach number increases as the soliton travels into a background of increasing density, or, equivalently, decreasing Alfven speed. Due to the simplicity of this formulation, which contains only one nonuniformity parameter, we are unable to pin down whether the characteristic wave speed is the sole parameter that controls the change of the Mach number, or whether other background quantities also play some role. Our expectation for the characteristic wave speed to be the control parameter for changing Mach number arises from the common notion of wave steepening in shallow water waves as they approach the sea shore. This phenomenon is often understood as a result of a finite-amplitude disturbance rushing into a background that supports slower waves; conservation of the wave action forces the disturbance to become steeper. To resolve this issue, perhaps an extension to a finite-temperature plasma, where an additional nonuniformity parameter is present, will help.

Other than the above considerations, we also find in general that the soliton energy, momentum, and mass are not conserved as the soliton travels through an inhomogeneous background. This implies that the soliton can exchange energy, momentum, and mass with the background plasmas. Only in the small-amplitude limit, where the soliton looks like the KdV type, can the soliton energy be conserved in an inhomogeneous environment. However, despite the energy conservation, the soliton Mach number in this regime may still change as the environments change.

There are some interesting and practical extensions to which the present formulation may apply. We may consider a nonstatic background, say, containing a large-scale wave, and look into how the soliton responds to the wave. If we ignore the back-reaction of the soliton to the large-scale wave but concentrate on how the wave affects the soliton, the Hamiltonian ΔH will be time dependent and, therefore, much like the dynamics of a classical particle, the soliton trajectory will become chaotic and unpredictable. At a more sophisticated level, one may also examine how a collection of solitons react back to the large-scale wave. Can a phenomenon analogous to the (inverse) Landau damping for a collection of collisionless particles ever occur in a collection of solitons? One may ask the following question: mediated by wave emission and absorption, is it possible for the soliton-wave system to reach some kind of equilibrium distribution that has been so profoundly established in classical mechanics at the turn of the 20th century?

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APPENDIX: ASSOCIATION OF CHARGE SEPARATION WITH THE $\omega_{ce} \gg \omega_{pe}$ REGIME

For simplicity and clarity, we will present a linear analysis that demonstrates the associations of charge separation with the regime $\omega_{ce} \gg \omega_{pe}$, and of charge quasineutrality with the opposite regime. The latter regime is much more commonly considered in standard plasma physics textbooks in connection with the lower hybrid waves, and the former can actually be shown to yield a charged ion wave with its dispersion relation given immediately below Eq. (25).

The linearized ion equations are

$$\omega \delta n_i = k n_0 \delta u_i , \tag{A1}$$

$$-i m_i \omega \delta u_i = e(\delta E_x + \delta v_i B_0 / c), \tag{A2}$$

and

$$-i m_i \omega \delta v_i = e(\delta E_y - \delta u_i B_0 / c). \tag{A3}$$

The linearized electron equations are

$$\omega \delta n_e = k n_0 \delta u_e , \tag{A4}$$

$$i m_e \omega \delta u_e = e(\delta E_x + \delta v_e B_0 / c), \tag{A5}$$

$$i m_e \omega \delta v_e = e(\delta E_y - \delta u_e / B_0 / c). \tag{A6}$$

The Maxwell's equations read

$$k \delta E_x = -i 4 \pi e(\delta n_i - \delta n_e), \tag{A7}$$

$$\omega \delta B = k c \delta E_y , \tag{A8}$$

$$k c \delta B = i 4 \pi e n_0 (\delta v_i - \delta v_e). \tag{A9}$$

Here we have assumed that $k = k_x$, $\delta B = \delta B_z$, and $\omega \ll kc$, the perpendicularly propagating compressional waves.

Subtracting Eq. (A1) from Eq. (A4), adding Eq. (A2) to Eq. (A5), and combining the resulting two equations with the Poisson's equation, Eq. (A7), we obtain a relation between δE_x and the transverse particle momenta:

$$\delta E_x = \frac{4 \pi n_0 c}{B_0} (m_i \delta v_i + m_e \delta v_e), \tag{A10}$$

the linear counterpart of Eq. (15).

Next, it is noted that in both lower-hybrid and charged ion-wave cases, the electron inertia in the x direction can be ignored. When we do so, δv_e can be solved from Eq. (A5),

$$\delta v_e = -\frac{c \delta E_x}{B_0}, \tag{A11}$$

which can be substituted into Eq. (A10) to yield a relation between δv_i and δv_e:

$$m_i \delta v_i = -\left(1 + \frac{\omega_{ce}^2}{\omega_{pe}^2}\right) m_e \delta v_e . \tag{A12}$$

It is important to note the appearance of the factor $\omega_{ce}^2 / \omega_{pe}^2$ in the above expression. When this factor is much smaller than unity, i.e., weakly magnetized electrons, we find that $m_i \delta v_z + m_e \delta v_e \approx 0$; upon referring to Eq. (A10), we obtain the charge quasineutrality, $\delta E_x \approx 0$. On the other hand, when the electrons are strongly magnetized, $\omega_{ce}^2 / \omega_{pe}^2 \gg 1$, we find that $m_i \delta v_i \approx (\omega_{ce}^2 / \omega_{pe}^2) m_e \delta v_e \gg m_e \delta v_e$; the ion transverse momentum overwhelmingly dominates the electron transverse momentum, and

$$\delta E_x \approx \left(\frac{4 \pi n_0 c}{B_0}\right) m_i \delta v_i . \tag{A13}$$

The above linear analysis has already made it clear as to why the parameter $\omega_{ce} / \omega_{pe}$ is so crucial in differentiating the present charge-separated solitons from the conventional charge-neutral solitons.

To find the corresponding dispersion relation for each regime, we have to consider the electromagnetic effects through δE_y. Adding Eq. (A3) to Eq. (A6), and then substituting the resulting current and δE_y from Eq. (A8) into the Ampere's law, Eq. (A9), we find a relation between δE_y and the longitudinal particle momenta,

$$\delta E_y = \frac{\omega^2}{k^2 c^2} \left(\frac{4 \pi n_0 c}{B_0}\right) (m_i \delta u_i + m_e \delta u_e). \tag{A14}$$

In fact, the x component of the electron inertia has already been ignored previously to obtain Eq. (A11), and hence the electron contribution to the right side of Eq. (A14) should also be ignored. This yields a relation between δE_y and δu_i.

The desired dispersion relations can finally be determined by using the above results and solving the two coupled equations for the ion x and y momenta. It is straightforward to obtain that

$$\frac{\omega^2}{k^2} = \frac{V_A^2}{1 + (k^2 V_A^2 / \omega_{pi}^2)} \tag{A15}$$

for $\omega_{ce} \gg \omega_{pe}$, and

$$\frac{\omega^2}{k^2} = \frac{V_A^2}{1 + (k^2 c^2 / \omega_{pe}^2)} \tag{A16}$$

for $\omega_{ce} \ll \omega_{pe}$. Equation (A16) is a familiar dispersion relation, describing the lower-hybrid waves, and Eq. (A15) is our finding in the strongly magnetized regime. The above analysis shows the essential differences, even in the linear regime, between the strongly-magnetized-electron case and the weakly-magnetized-electron case.

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